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New Decoupled Framework for Reliability-Based Design Optimization

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Traditionally, reliability-based design optimization (RBDO) has been formulated as a nested optimization problem. The inner loop, generally, involves the solution to optimization problems for computing the probabilities of failure of the critical failure modes, and the outer loop performs optimization by varying the decision variables. Such formulations are by nature computationally intensive, requiring numerous function and constraint evaluations. To alleviate this problem, researchers have developed iterative decoupled RBDO approaches. These methods perform deterministic optimization and reliability assessment in a sequential manner until a consistent reliability-based design is obtained. The sequential methods are attractive because a consistent reliable design can be obtained at considerably lower computational cost. However, the designs obtained by using these decoupled approaches cannot guarantee production of the true solution. A new decoupled method for RBDO is developed in this investigation. Postoptimal sensitivities of the most probable point (MPP) of failure with respect to the decision variables are introduced to update the MPPs during the deterministic optimization phase of the proposed approach. A damped Broyden–Fletcher–Goldfarb–Shanno method is used to significantly reduce the cost of obtaining these sensitivities. It is the use of postoptimal sensitivities that differentiates this new decoupled RBDO approach from previous efforts.

Nomenclature

$\boldsymbol{d}$ = design variables
$\boldsymbol{d}^l$ = lower bounds on design variables
$\boldsymbol{d}^u$ = upper bounds on design variables
f = merit function
$\boldsymbol{g}^D$ = deterministic constraints
$\boldsymbol{g}^R$ = failure modes or probabilistic hard constraints
$\boldsymbol{g}^{\mathrm{RC}}$ = reliability constraints
N_{hard} = number of hard constraints
N_{soft} = number of soft constraints
P_{allow_i} = allowable probability of failure for ith failure mode
P_i = probability of failure of ith failure mode
$\boldsymbol{p}$ = fixed parameters in deterministic optimization
$\boldsymbol{U}$ = standard normal random variables
$\boldsymbol{u}^*$ = most probable point in $\boldsymbol{U}$ space
$\boldsymbol{X}$ = random variables
$\boldsymbol{x}^*$ = most probable point in $\boldsymbol{X}$ space
$\boldsymbol{Y}$ = random state variables
$\boldsymbol{y}$ = deterministic state variables
β_i = reliability index of ith failure mode
β_{reqd_i} = desired reliability index of ith failure mode
$\boldsymbol{\eta}$ = limit-state parameters
$\boldsymbol{\theta}$ = distribution parameters

I. Introduction

IN a deterministic design optimization, the designs are often driven to the limit of the design constraints, leaving little or no latitude for uncertainties. The resulting deterministic optimum is usually associated with a high probability of failure of the artifact being designed, due to the influence of uncertainties inherently present during the modeling and manufacturing phases of the artifact and due to uncertainties in the external operating conditions of the artifact. The uncertainties include variations in certain parameters, which are either controllable, for example, dimensions, or uncontrollable, for example, material properties, and model uncertainties and errors associated with the simulation tools used for simulation-based design.[1,2] In this investigation, variational uncertainties are modeled as continuous random variables. Other forms of uncertainty, such as model uncertainties and errors associated with simulation tools, are assumed to be minimal, and it is assumed that the analysis tools can reasonably predict the actual performance behavior.

Optimized deterministic designs determined without considering the uncertainties can be unreliable and might lead to catastrophic failure of the artifact being designed. Uncertainties in simulation-based design are inherently present and need to be accounted for in the design optimization process. Reliability-based design optimization (RBDO) is a methodology that addresses this problem. In designing artifacts with multiple failure modes, it is important that an artifact be designed such that it is sufficiently reliable with respect to each of the critical failure modes or to the overall system failure. The reliability index, or the probability of failure corresponding to either a failure mode or the system, can be computed by performing a reliability analysis. In a RBDO formulation, the critical failure modes in deterministic optimization are replaced with constraints on probabilities of failure corresponding to each of the failure driven modes or with a single constraint on the system probability of failure.[3,4]

Traditionally, researchers have formulated RBDO as a nested optimization problem (also known as a double-loop formulation). Such a formulation is, by nature, computationally expensive because of the inherent computational expense required for the reliability analysis, which itself involves the solution to an optimization problem. Solving such nested optimization problems is cost prohibitive, especially for large-scale multidisciplinary systems, which are themselves computationally intensive. Moreover, the computational cost associated with RBDO grows exponentially as the number of random variables and the number of critical failure modes increase. To alleviate the high computational cost, researchers have developed iterative methodologies for RBDO. In these methodologies, a deterministic optimization and a reliability analysis are performed sequentially, and the procedure is repeated until desired convergence is achieved. Such techniques are referred to as sequential RBDO techniques in the rest of the paper. The sequential RBDO techniques offer a practical means of obtaining a reliable design at considerably reduced computational cost. However, the designs

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obtained using existing sequential RBDO methods cannot be guaranteed to produce a true most probable point (MPP) solution. In this investigation, a new sequential method for reliability-based design optimization is developed. In the proposed method, the sensitivities of the MPP of failure with respect to the decision variables are computed to update the MPPs during the deterministic optimization phase of the proposed sequential RBDO approach. The MPP update is based on the first-order Taylor series expansion around the design point from the preceding cycle. Postoptimality methods combined with a damped Broyden–Fletcher–Goldfarb–Shanno (BFGS) update scheme are used to evaluate these sensitivities at significantly reduced cost. The MPP update is found to be extremely accurate, especially in the vicinity of the point from the preceding cycle. The methodology not only finds the true optimal solution but also the exact MPPs of failure, which is important to ensure that the target reliability index is satisfied.

Note that in the sequential RBDO method by Wang and Kodiyalam,[5] the MPP of failure for each failure-driven constraint is estimated approximately. If the limit-state function in the standard space in sufficiently nonlinear, which is often the case, then the MPP estimates might be extremely inaccurate, which might result in a spurious optimal design. In Ref. 6, the sequential optimization and reliability assessment (SORA) methodology is developed, and an exact first-order reliability analysis is performed to obtain the MPP of failure for each failure-driven constraint. Therefore, a reliable design that satisfies a probability of failure requirement is guaranteed to be obtained using this framework. However, a true local optimum cannot be guaranteed. This is because the MPP of failure for the hard constraints are obtained at the preceding design point. A shift factor $\boldsymbol{s}_i$ from the mean values of the random variables is calculated and is used to update the MPP of failure for probabilistic constraint evaluation during the deterministic optimization phase in the next iteration because the optimizer varies the mean values of the random variables. This MPP update might be inaccurate because as the optimizer varies the design variables, the MPP of failure (and, hence, the shift factor) also changes and is not addressed in SORA. This might lead to pseudooptimum solutions.

In this paper, the use of postoptimality analysis combined with a damped BFGS update strategy allows the MPP to be approximated linearly in the deterministic optimization as the design variables are changed. The postoptimality analysis provides sensitivities of how the optimal value of the MPP changes with changes in the design variables. The use of postoptimality analysis has a long history of use in nested multilevel optimization frameworks.[7–10] Postoptimal sensitivities are used to account for couplings when employing sequential decoupled analysis techniques. In each iteration of the algorithm, postoptimality analysis is used to account for MPP coupling. The sequential nature of the algorithm and that the algorithm approaches the true MPP iteratively ensures that that the linear approximation is sufficient at convergence.

Note that solution of the MPP for a candidate design involves solution to an optimization problem. [See, for example, Eqs. (21) and (22).] Postoptimality techniques are used to evaluate the optimal sensitivities of the MPP with respect to the design variables, which are fixed parameters in the MPP optimization. In this paper a series of reliability-based design test problems are employed to illustrate the benefits of the proposed framework.

This paper is organized as follows. In Sec. II, a brief description of a traditional deterministic design optimization is given. In Sec. III, an overview of the existing RBDO formulations and reliability-analysis techniques is presented. In Sec. IV, a new sequential RBDO methodology is presented, followed by numerical results in Sec. V.

II. Deterministic Design Optimization

In a deterministic design optimization, the designer seeks the optimum values of design variables for which the merit function is the minimum and the deterministic constraints are satisfied. A typical deterministic design optimization problem can be formulated as follows. Minimize

$$f(\boldsymbol{d},\boldsymbol{p}) \tag{1}$$

subject to

$$g_i^R(\boldsymbol{d},\boldsymbol{p}) \geq 0, \qquad i = 1,\ldots,N_{\text{hard}} \tag{2}$$

$$g_j^D(\boldsymbol{d},\boldsymbol{p}) \geq 0, \qquad j = 1,\ldots,N_{\text{soft}} \tag{3}$$

$$\boldsymbol{d}^l \leq \boldsymbol{d} \leq \boldsymbol{d}^u \tag{4}$$

where $\boldsymbol{d}$ are the design variables and $\boldsymbol{p}$ are the fixed parameters of the optimization problem. Here, g_i^R is the ith hard constraint that models the ith critical failure mechanism of the system, for example, stress, deflection, and loads, and g_j^D is the jth soft constraint that models the jth deterministic constraint due to other design considerations, for example, cost and marketing. The design space is bounded by $\boldsymbol{d}^l$ and $\boldsymbol{d}^u$. The merit function and the constraints are explicit functions of $\boldsymbol{d}$ and $\boldsymbol{p}$. If $g_i^R < 0$ at a given design $\boldsymbol{d}$, the artifact is said to have failed with respect to the ith failure mode. Though a clear distinction is made between hard and soft constraints, deterministic design optimization treats both these types of constraints similarly, and the failure of the artifact due to the presence of uncertainties is not taken into consideration.

A deterministic optimization formulation does not account for the uncertainties in the design variables and parameters. Optimized designs based on a deterministic formulation are usually associated with a high probability of failure because of the likely violation of certain hard constraints in service. This is particularly true if the hard constraints are active at the deterministic optimum solution. In today's competitive marketplace, it is very important that the resulting designs are optimal as well as reliable. This is usually achieved by replacing a deterministic optimization formulation with an RBDO formulation, where the critical hard constraints are replaced with reliability constraints.

III. RBDO

Modern competitive market demands have forced the designers to develop techniques for obtaining optimized designs that are also reliable. In the past two decades, researchers have proposed a variety of frameworks for efficiently performing RBDO. A detailed description of the various RBDO frameworks is given in the following sections.

A. Double-Loop Methods for RBDO

Traditionally, the reliability-based optimization problem has been formulated as a double-loop optimization problem. In a typical RBDO formulation, the critical hard constraints from the deterministic formulation are replaced by reliability constraints, as minimize

$$f(\boldsymbol{d},\boldsymbol{p}) \tag{5}$$

subject to

$$\boldsymbol{g}^{\text{RC}}(\boldsymbol{X},\boldsymbol{\eta}) \geq 0 \tag{6}$$

$$g_j^D(\boldsymbol{d},\boldsymbol{p}) \geq 0, \qquad j = 1,\ldots,N_{\text{soft}} \tag{7}$$

$$\boldsymbol{d}^l \leq \boldsymbol{d} \leq \boldsymbol{d}^u \tag{8}$$

where $\boldsymbol{g}^{\text{RC}}$ are the reliability constraints. They are either constraints on probabilities of failure corresponding to each hard constraint (component formulation) or are a single constraint on the overall system probability of failure (system formulation). In this paper, only component failure modes are considered. Note that the reliability constraints depend on the random variables $\boldsymbol{X}$ and limit-state parameters $\boldsymbol{\eta}$. The distribution parameters of the random variables are obtained from the design variables $\boldsymbol{d}$ and the fixed parameters $\boldsymbol{p}$. (See the section on reliability analysis hereafter.) Now, $\boldsymbol{g}^{\text{RC}}$ can be formulated as

$$g_i^{\text{RC}} = P_{\text{allow}_i} - P_i, \qquad i = 1,\ldots,N_{\text{hard}} \tag{9}$$

where P_i is the failure probability of the hard constraint g_i^R at a given design and P_{allow_i} is the allowable probability of failure for this failure mode. The probability of failure is usually estimated by employing standard reliability techniques. A brief description of standard reliability methods is given in the next section. Note that the RBDO formulation just given [Eqs. (5–8)] assumes that the violation of soft constraints due to variational uncertainties are permissible and can be traded off for more reliable designs. For practical problems, design robustness represented by the merit function and the soft constraints could be a significant issue, one that would require a hybrid robustness and RBDO formulation.

B. Reliability Analysis

Reliability analysis is a tool to compute the reliability index or the probability of failure corresponding to a given failure mode or for the entire system.[11] The uncertainties are modeled as continuous random variables, $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$, with known (or assumed) joint cumulative distribution function $F_X(\boldsymbol{x})$. The design variables $\boldsymbol{d}$ consist of either distribution parameters $\boldsymbol{\theta}$ of the random variables $\boldsymbol{X}$, such as means, modes, standard deviations, and coefficients of variation, or deterministic parameters, also called limit-state parameters, denoted by $\boldsymbol{\eta}$. The design parameters $\boldsymbol{p}$ consist of either the means, the modes, or any first-order distribution quantities of certain random variables. Mathematically, this can be represented by the statements

$$[\boldsymbol{p}, \boldsymbol{d}] = [\boldsymbol{\theta}, \boldsymbol{\eta}] \tag{10}$$

$$\boldsymbol{p} \text{ is a subvector of } \boldsymbol{\theta} \tag{11}$$

Random variables can be consistently denoted as $\boldsymbol{X}(\boldsymbol{\theta})$, and the ith failure mode can be denoted as $g_i^R(\boldsymbol{X}, \boldsymbol{\eta})$. In the following text, $\boldsymbol{x}$ denotes a realization of the random variables $\boldsymbol{X}$, and the subscript i is dropped without loss of clarity. When $g^R[\boldsymbol{x}(\boldsymbol{\theta}), \boldsymbol{\eta}] \leq 0$ represents the failure domain, and $g^R[\boldsymbol{x}(\boldsymbol{\theta}), \boldsymbol{\eta}] = 0$ is the so-called limit-state function, the time-invariant probability of failure for the hard constraint is given by

$$P(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_{g^R(x,\eta) \leq 0} f_X(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{12}$$

where $f_X(\boldsymbol{x})$ is the joint probability density function of $\boldsymbol{X}$. It is usually impossible to find an analytical expression for the preceding integral. In standard reliability techniques, a probability distribution transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is usually employed. An arbitrary n-dimensional random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)^T$ is mapped into an independent standard normal vector $\boldsymbol{U} = (U_1, U_2, \ldots, U_n)^T$. This transformation is known as the Rosenblatt transformation.[12] The standard normal random variables are characterized by a zero mean and unit variance. The limit-state function in $\boldsymbol{U}$ space can be obtained as $g^R[\boldsymbol{x}, \boldsymbol{\eta}] = g^R[T^{-1}(\boldsymbol{u}), \boldsymbol{\eta}] = G^R(\boldsymbol{u}, \boldsymbol{\eta}) = 0$. The failure domain in $\boldsymbol{U}$ space is $G^R(\boldsymbol{u}, \boldsymbol{\eta}) \leq 0$. Equation (12), thus, transforms to

$$P(\boldsymbol{\theta}, \boldsymbol{\eta}) = \int_{G^R(u,\eta) \leq 0} \phi_U(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \tag{13}$$

where $\phi_U(\boldsymbol{u})$ is the standard normal density. If the limit-state function in $\boldsymbol{U}$ space is affine, that is, if $G^R(\boldsymbol{u}, \boldsymbol{\eta}) = \boldsymbol{\alpha}^T \boldsymbol{u} + \beta$, then an exact result for the probability of failure is $P_f = \Phi[-(\beta/\|\boldsymbol{\alpha}\|)]$, where $\Phi(\,)$ is the cumulative Gaussian distribution function. If the limit-state function is close to being affine, that is, if $G^R(\boldsymbol{u}, \boldsymbol{\eta}) \approx \boldsymbol{\alpha}^T \boldsymbol{u} + \beta$ with $\beta = -\boldsymbol{\alpha}^T \boldsymbol{u}^*$, where $\boldsymbol{u}^*$ is the solution of the following optimization problem, then minimize

$$\|\boldsymbol{u}\| \tag{14}$$

subject to

$$G^R(\boldsymbol{u}, \boldsymbol{\eta}) = 0 \tag{15}$$

and then the first-order estimate of the probability of failure is $P_f = \Phi[-(\beta/\|\boldsymbol{\alpha}\|)]$, where $\boldsymbol{\alpha}$ represents a normal to the manifold (15) at the solution point. The solution $\boldsymbol{u}^*$ of the preceding optimization problem, the so-called design point, β point, or the MPP of failure, defines the reliability index $\beta_p = -(\boldsymbol{\alpha}^T \boldsymbol{u}^*/\|\boldsymbol{\alpha}\|)$. This method of estimating the probability of failure is known as the first-order reliability method (FORM).[11]

When the FORM estimate is used, the reliability constraints in Eq. (9) can be written in terms of reliability indices as

$$g_i^{\mathrm{RC}} = \beta_i - \beta_{\mathrm{reqd}_i} \tag{16}$$

where β_i is the first-order reliability index and $\beta_{\mathrm{reqd}_i} = -\Phi^{-1}(P_{\mathrm{allow}_i})$ is the desired reliability index for the ith hard constraint. When the reliability constraints are formulated as given in Eq. (16), the approach is referred to as the reliability index approach (RIA).

A variety of specialized algorithms have been reported in the literature to solve the FORM problem.[13] The Hasofer–Lind and Rackwitz–Fiessler (HL–RF) algorithm based on the Newton–Raphson root solving approach is very popular in the structural reliability community. Variants to the HL–RF algorithm exist that use additional line searches. These algorithms may fail to converge for highly nonlinear or badly scaled problems because they are based on the first-order approximations of the limit-state functions. A sequential quadratic programming (SQP) algorithm is often a more robust algorithm. The solution typically requires many system analysis evaluations. Moreover, there might be cases where the optimizer may fail to provide a solution to the FORM problem, especially when the limit-state surface is far away from the origin in $\boldsymbol{U}$ space or when the case $G^R(\boldsymbol{u}, \boldsymbol{\eta}) = 0$ never occurs at a particular design setting. In other words, RIA is associated with singularity.[14]

In design automation, it cannot be known a priori what design points the upper-level optimizer will visit; therefore, it is not known if the optimizer for the FORM problem will provide a consistent result. This problem was addressed recently by Padmanabhan and Batill[15] by using a trust region algorithm for equality constrained problems. For cases when $G^R(\boldsymbol{u}, \boldsymbol{\eta}) = 0$ does not occur, the algorithm provided the best possible solution for the problem through minimizing

$$\|\boldsymbol{u}\| \tag{17}$$

subject to

$$G^R(\boldsymbol{u}, \boldsymbol{\eta}) = c \tag{18}$$

The reliability constraints formulated by the RIA are, therefore, not robust. RIA is usually more effective if the probabilistic constraint is violated, but it yields a singularity if the design has a failure probability of zero or one.[14] To overcome this difficulty, Tu et al.[14] provided an improved formulation to prescribe the probabilistic constraint. In this method, known as the performance measure approach (PMA), the reliability constraints are stated by an inverse formulation as

$$g_i^{\mathrm{RC}} = G_i^R\left(\boldsymbol{u}^*_{i_{\beta=\rho}}, \boldsymbol{\eta}\right) \tag{19}$$

where $\boldsymbol{u}^*_{i_{\beta=\rho}}$ is the solution to the inverse reliability analysis (IRA) optimization problem, minimizing

$$G_i^R(\boldsymbol{u}, \boldsymbol{\eta}) \tag{20}$$

subject to

$$\boldsymbol{u}^T \boldsymbol{u} = \rho^2 = \beta^2_{\mathrm{reqd}_i} \tag{21}$$

where the optimum solution $\boldsymbol{u}^*_{i_{\beta=\rho}}$ corresponds to MPP in IRA of the ith hard constraint. Solving RBDO by the PMA formulation is usually more efficient and robust than the RIA formulation where the reliability is evaluated directly. The efficiency lies in that the search for the MPP of an inverse reliability problem is easier to solve than the search for the MPP corresponding to an actual reliability. The RIA and the PMA approaches for RBDO are essentially inverse of one another and would yield the same solution if the constraints are active at the solution.[14] If the constraint on the reliability index (as in the RIA formulation) or the constraint on the optimum value of the limit-state function (as in the PMA formulation) is not active at the solutions, the reliable solutions obtained from the two approaches might differ.

Similar RBDO formulations were independently developed by other researchers.[16–18] In these RBDO formulations, constraint (21) is considered as an inequality constraint ($\boldsymbol{u}^T \boldsymbol{u} \leq \beta^2_{\mathrm{reqd}_i}$), which is a robust way of handling the constraint on the reliability index. The major difference lies in that in these investigations semi-infinite optimization algorithms were employed to solve the RBDO problem. However, the overall RBDO was still a nested optimization procedure.

Note that irrespective of the formulation used to prescribe the reliability constraints (RIA or PMA), the traditional RBDO formulation involves a nested optimization procedure. Each iteration of RBDO requires the evaluation of the reliability constraints, which themselves require solution to optimization problems. Such a formulation is inherently computationally intensive. Moreover, the formulation becomes computationally intractable as the number of hard constraints increase, which is often the case in most real-life design problems. To address this issue, a variety of sequential RBDO methodologies have been developed.

C. Sequential RBDO Methods

The basic concept behind sequential RBDO techniques is to decouple the upper-level optimization from the reliability analysis to avoid a nested optimization problem. In sequential RBDO methods, the main optimization and the search of the MPPs of failure (reliability analysis) is performed separately, and the procedure is repeated until desired convergence is achieved. The idea is to find a consistent reliable design at considerably lower computational cost as compared to the nested approach. A consistent reliable design is a feasible design that satisfies all of the reliability constraints and other soft constraints. The reliability analysis is used to check whether a given design meets the desired reliability level. In most sequential techniques of RBDO, a design obtained by performing a deterministic optimization is updated based on the information obtained from the reliability analysis or by using some nonlinear transformations, and the updated design is used as a starting point for the next iteration. Some of these techniques are discussed hereafter for the sake of completeness.

Chen et al.[19] proposed a single-loop RBDO methodology for normally distributed random variables. Wang and Kodiyalam[5] generalized this methodology for nonnormal random variables and reported enormous computational savings when compared to the nested RBDO formulation. Instead of using the reliability analysis (inverse reliability problem) to obtain the true MPP of failure ($\boldsymbol{u}^*_{\beta=\rho}$), Wang and Kodiyalam use the direction cosines of the probabilistic constraint at the mean values of the random variables in the standard space $[\boldsymbol{\alpha} = (\partial g^R/\partial \boldsymbol{u})/\|\partial g^R/\partial \boldsymbol{u}\|]$ and the target reliability index ρ to make an estimate of the MPP of failure ($\tilde{\boldsymbol{u}}^*_{\beta=\rho} = -\rho\boldsymbol{\alpha}$) (Fig. 1). Note that the estimated MPPs lie on the target reliability sphere. During optimization the corresponding MPP in $\boldsymbol{X}$ space needs to be calculated to evaluate the probabilistic performance functions. The MPP of failure in $\boldsymbol{X}$ space is found by mapping $\tilde{\boldsymbol{u}}^*_{\beta=\rho}$ to the original space. If the random variables in $\boldsymbol{X}$ space are independent and normally distributed, then the MPP in original space is given by $\tilde{\boldsymbol{x}}^* = \mu_x - \tilde{\boldsymbol{u}}^*_{\beta=\rho}\sigma_x$. If the variables have a nonnormal distribution, then the equivalent means μ'_x and equivalent standard deviations σ'_x of an approximate normal distribution are computed and used in the preceding expression to estimate the MPP in $\boldsymbol{X}$ space. A detailed description of this approach can be found in Ref. 5.

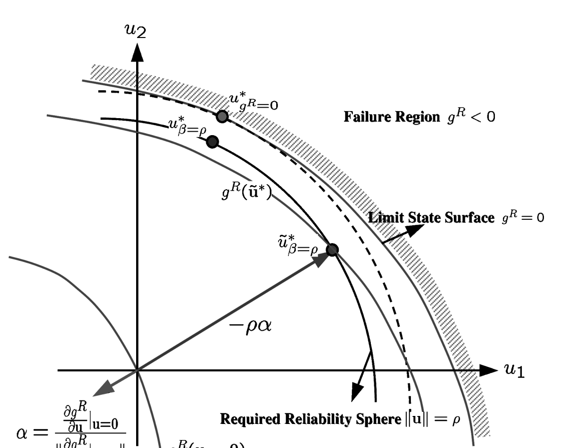

Fig. 1 Approximate MPP estimation.

The advantage of this methodology is that it completely eliminates the lower-level optimization for evaluating the reliability-based constraints. The MPP of failure for each failure-driven constraint is estimated approximately. If the limit-state function is close to linear in the standard space, then the estimate of the MPP in $\boldsymbol{U}$ space will be accurate enough, and the final solution may be close to the actual solution. However, if the limit-state function in the standard space in sufficiently nonlinear, which is often the case in most real-life design problems, then the MPP estimates might be extremely inaccurate, which might result in a spurious optimal design.

Chen and Du[6] also proposed a SORA methodology. In SORA, the boundaries of the violated constraints (with low reliability) are shifted into the feasible direction based on the reliability information obtained in the preceding iteration. Two different formulations were used for reliability assessment, the probability formulation (RIA) and the percentile performance formulation (PMA). The percentile formulation was reported to be computationally less demanding compared to the probability formulation, and the overall cost of RBDO was reported to be significantly less compared to the nested formulation. Note that in this methodology an exact first-order reliability analysis is performed to obtain the MPP of failure for each failure-driven constraint, which was not the case in the approximate RBDO methodology of Wang and Kodiyalam.[5] Therefore, a consistent reliable design is almost guaranteed to be obtained from this framework. However, a true local optimum cannot be guaranteed. This is because the MPP of failure for the hard constraints are obtained at the preceding design point. A shift factor $\boldsymbol{s}_i$ from the mean values of the random variables is calculated and is used to update the MPP of failure for probabilistic constraint evaluation during the deterministic optimization phase in the next iteration because the optimizer varies the mean values of the random variables. This MPP update might be inaccurate because as the optimizer varies the design variables, the MPP of failure (and, hence, the shift factor) also changes and is not addressed in SORA. This might lead to pseudo-optimum solutions.

In this paper, a new decoupled SORA strategy is being developed. The sensitivities of the MPPs with respect to the design variables are used to update them during the deterministic optimization phase. This helps in a good estimate of the MPP as the design variables are varied by the optimizer during the deterministic optimization phase. The proposed methodology is detailed in the following section.

IV. Proposed RBDO Methodology

In this investigation, a framework for efficiently performing RBDO is developed. As described earlier, a traditional nested RBDO formulation is extremely impractical for most real-life design problems of reasonable size and scope. Researchers have proposed sequential RBDO approaches to speed up the optimization process and, therefore, obtain a consistent reliability-based design. These approaches are practical and attractive because a workable design can be obtained at considerably lower computational cost. In the following text, a new sequential RBDO methodology is described.

The flowchart of the proposed RBDO methodology is shown in Fig. 2. The methodology consist of the following steps:

1) Given an initial design $\boldsymbol{d}^0$, set iteration counter $k = 0$.

2) Solve the following deterministic optimization problem starting from design $\boldsymbol{d}^k$ to get a new design $\boldsymbol{d}^{k+1}$: minimizing (with respect to d)

$$f(\boldsymbol{d}, \boldsymbol{p}) \tag{22}$$

subject to

$$g_i^R\left(\boldsymbol{x}_i^*, \boldsymbol{\eta}\right) \geq 0, \qquad i = 1, \ldots, N_{\text{hard}} \tag{23}$$

$$g_j^D(\boldsymbol{d}, \boldsymbol{p}) \geq 0, \qquad j = 1, \ldots, N_{\text{soft}} \tag{24}$$

$$\boldsymbol{d}^l \leq \boldsymbol{d} \leq \boldsymbol{d}^u \tag{25}$$

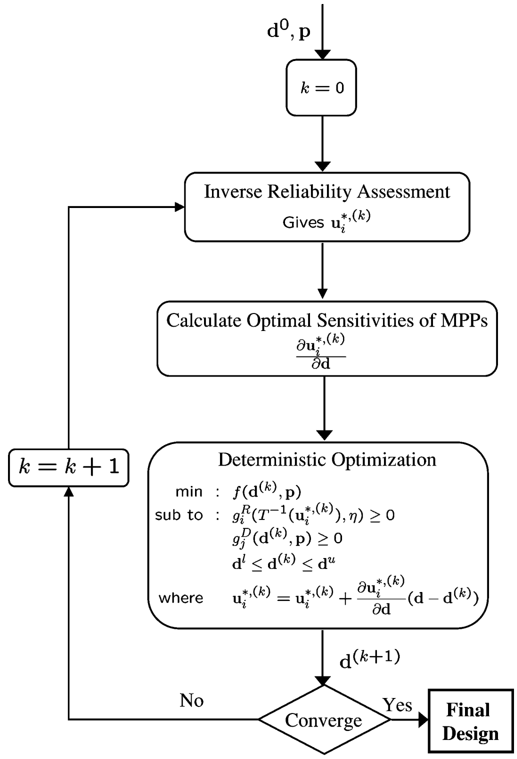

Fig. 2 Proposed RBDO methodology.

During the first iteraton, $k = 0$, the MPP of failure for evaluating the probabilistic constraints is set equal to the mean values of the random variables, $\boldsymbol{x}_i^* = \mu_x$. Note that the mean of the random variables (distribution parameters) is a subset of the design variables and fixed parameters [Eq. (10)]. This corresponds to solving the deterministic optimization problem, Eqs. (1–4). From the authors' experience, it has been observed that starting from a deterministic solution results in lower computational cost for RBDO. In subsequent iterations, $k > 0$, the MPP of failure for evaluating the probabilistic constraints is obtained from the first-order Taylor series expansion about the preceding design point:

$$\boldsymbol{u}_i^* = \boldsymbol{u}_i^{*,k} + \frac{\partial \boldsymbol{u}_i^{*,k}}{\partial \boldsymbol{d}}(\boldsymbol{d} - \boldsymbol{d}^k), \qquad i = 1, \ldots, N_{\text{hard}} \tag{26}$$

Note that $\partial \boldsymbol{u}_i^{*,k}/\partial \boldsymbol{d}$ is a matrix and its columns contain the gradient of the MPPs with respect to each of the decision variables. For example, the first column of the matrix contains the gradient of the MPP vector $\boldsymbol{u}_i^*$ with respect to the first design variable d_1. The MPPs in the $\boldsymbol{X}$ space are obtained by using the following transformation.

3) Check for convergence on the design variables and the MPPs and that the constraints are satisfied. If converged, stop. Else, go to the next step.

4) At the new design $\boldsymbol{d}^{k+1}$, perform an exact first-order IRA [Eqs. (20–21)] for each hard constraint. This gives the MPP of failure of each hard constraint, $\boldsymbol{u}_i^{*,k+1}$.

5) Compute the postoptimal sensitivities $\partial \boldsymbol{u}_i^{*,k}/\partial \boldsymbol{d}$ for each hard constraint, that is, how the MPP of failure will change with a change in design variables (See following section).

6) Set $k = k + 1$ and go to step 2.

A. Sensitivity of Optimal Solution to Problem Parameters

The proposed RBDO framework requires the optimal sensitivities of the MPPs with respect to the design variables. The postoptimal sensitivities are needed to update the MPPs based on linearization around the preceding design point. The following techniques could be used to compute the postoptimal sensitivities for the MPPs.

1) The sensitivity of the optimal solution to problem parameters can be computed by differentiating the first-order Karush–Kuhn–Tucker (KKT) optimality conditions (see Ref. 7). The Lagrangian L for the inverse reliability optimization problem [Eqs. (20–21)] is

$$L = G^R(\boldsymbol{u}, \boldsymbol{\eta}) + \lambda(\boldsymbol{u}^T \boldsymbol{u} - \rho^2) \tag{27}$$

where λ is the lagrange multiplier corresponding to the equality constraint, for example, h^R. The first-order optimality conditions for this problem are

$$\frac{\partial L}{\partial \boldsymbol{u}} = \frac{\partial G^R}{\partial \boldsymbol{u}^*} + 2\lambda \boldsymbol{u}^* = 0 \tag{28}$$

When the first-order KKT optimality conditions are differentiated with respect to a parameter in the vector, $\boldsymbol{z} = [\boldsymbol{\theta}, \boldsymbol{\eta}]^T$, the following linear system of equations is obtained:

$$\begin{bmatrix} \dfrac{\partial^2 L}{\partial \boldsymbol{u}^2} & \boldsymbol{u} \\ \boldsymbol{u}^T & 0 \end{bmatrix} \begin{pmatrix} \dfrac{\partial \boldsymbol{u}^*}{\partial z_l} \\ \dfrac{\partial \lambda^*}{\partial z_l} \end{pmatrix} + \begin{pmatrix} \dfrac{\partial^2 L}{\partial \boldsymbol{u} \partial z_l} \\ \dfrac{\partial h^R}{\partial z_l} \end{pmatrix} = 0 \tag{29}$$

The foregoing system needs to be solved for each parameter in the optimization problem to obtain the sensitivity of the optimal solution with respect to that parameter $\partial \boldsymbol{u}^*/\partial z_l$. In the proposed RBDO framework, the system of equations needs to be solved for only those parameters that are decision variables in the upper level optimization.

B. Damped BFGS Update

Note that the Hessian of the limit-state function needs to be computed when using this technique. If the Hessian of the limit-state function is not available or is difficult to obtain, other techniques have to be used. In the present implementation, a damped BFGS update is used to obtain the second-order information.[20] This method is defined by

$$r_k = \psi_k y_k + (1 - \psi_k) H_k s_k \tag{30}$$

where the scalar

$$\psi_k = \begin{cases} 1: & s_k^T y_k \geq 0.2 s_k^T H_k s_k \\ \dfrac{0.8 s_k^T H_k s_k}{s_k^T H_k s_k - s_k^T y_k}: & s_k^T y_k \leq 0.2 s_k^T H_k s_k \end{cases} \tag{31}$$

and y_k and s_k are the differences in the function and gradient values of the preceding iteration from the current iteration, respectively. The Hessian update is

$$H_{k+1} = H_k - \frac{H_k s_k s_k^T H_k}{s_k^T H_k s_k} + \frac{r_k r_k^T}{s_k^T r_k} \tag{32}$$

2) The sensitivity of optimal solution to problem parameters can also be obtained by using finite difference techniques. These techniques can be extremely expensive because the dimension of the decision variables and the number of hard constraints increase. This is because a full optimization is required to compute the sensitivity of the MPP with respect to each decision variable, and this has to be performed for each hard constraint. However, significant computational savings can be achieved if the preceding optimum MPP is used as a warm starting point to compute the change in MPP as the design variables are perturbed.

3) Approximations to the limit-state function can also be utilized to compute the sensitivity of an optimal solution to problem parameters. This technique is described hereafter.

a) At a given design $\boldsymbol{d}^k$, perform IRA to obtain the exact MPP $\boldsymbol{x}_i^{*,k}$.

b) Construct linear approximations of the hard constraint as follows:

$$\tilde{g}_i^R = g_i^R\left(\boldsymbol{x}_i^{*,k}, \boldsymbol{\eta}^k\right) + \left.\frac{\partial g_i^R}{\partial \boldsymbol{x}}\right|_{x_i^{*,k}, \eta^k}^T \left(\boldsymbol{x} - \boldsymbol{x}_i^{*,k}\right) + \left.\frac{\partial g_i^R}{\partial \boldsymbol{\eta}}\right|_{x_i^{*,k}, \eta^k}^T \left(\eta - \eta_i^k\right) \tag{33}$$

c) Perform IRA over the linear approximation at perturbed values of design variables to obtain approximate sensitivities.

V. Test Problems

The proposed decoupled RBDO methodology is tested using analytical, structural, and multidisciplinary design problems. The methodology is compared to the nested RBDO approach using the PMA approach for probabilistic constraint evaluation.

A. Short Rectangular Column

This problem has been used for testing and comparing RBDO methodologies by Kuschel and Rackwitz.[21] The design problem is to determine the depth h and width b of a short column with rectangular cross section with a minimal total mass bh assuming unit mass per unit area. The uncertain vector, $\boldsymbol{X} = (P, M, Y)$, the stochastic parameters, and the correlations of the vector elements are listed in Table 1. The limit-state function in terms of the random vector, $\boldsymbol{X} = (P, M, Y)$, and the limit-state parameters, $\boldsymbol{\eta} = (b, h)$, (which happen to be same as the design vector $\boldsymbol{d}$ in this problem) is given by

$$g^R(\boldsymbol{x}, \boldsymbol{\eta}) = 1 - 4M/bh^2Y - P^2/(bhY)^2 \tag{34}$$

The objective function is given by

$$f(\boldsymbol{d}) = bh \tag{35}$$

The depth h and the width b of the rectangular column had to satisfy $15 \leq h \leq 25$ and $5 \leq b \leq 15$. The allowable failure probability is 0.00621, or, in other words, a reliability index for the failure mode greater than or equal to 2.5. The optimization process was started from the point $(\boldsymbol{u}^0, \boldsymbol{d}^0) = [(1, 1, -1), (5, 15)]$. Both approaches result in an optimal solution $\boldsymbol{d}^* = (8.668, 25.0)$. The computational effort for this problem is compared in Table 2. The nested approach requires 77 evaluations of the limit-state function and 85 evaluations of its gradients as compared to 31 evaluations of the limit-state function and 31 evaluation of its gradients for the proposed framework. Therefore, note that the proposed methodology for RBDO is more computationally efficient than the traditional RBDO approach. The proposed method took three cycles for convergence, the design history for which is shown in Fig. 3. Note that after the first cycle, that is, iteration zero, the new design obtained is close to the optimum.

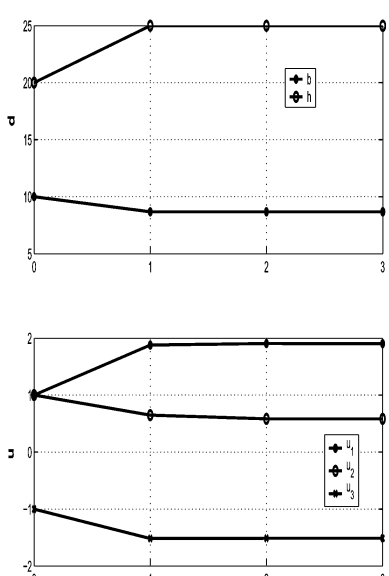

Fig. 3 Convergence history for example problem.

In Fig. 3 the initial design point, $\boldsymbol{d}^0 = (5, 15)$, before iteration zero is not shown. The deterministic optimization in iteration zero obtains the optimum $\boldsymbol{d}^0 = (10, 20)$ using the mean values of the uncertain parameters $\boldsymbol{u}^0 = (1, 1, -1)$. A cycle here refers to steps 1–3 of the algorithm. In Fig. 3 the values of $\boldsymbol{u}$ shown are the nominal MPP estimate used in the deterministic optimization for that iteration. As expected, the MPP is not converged after iteration zero. In decoupled methods the deterministic optimization cannot account for changes in the MPP due to changes in the design variables. In this new methodology, a measure of the coupling is restored, where changes in the MPP are linearly estimated using postoptimality analysis. The proposed methodology is able to converge to the true MPP within a few cycles.

B. Analytical Problem

This is an analytical multidisciplinary test problem. Even though the problem is only two-dimensional, it is sufficiently nonlinear and has the attributes of a general multidisciplinary problem. This problem has two design variables, d_1 and d_2, and two parameters, p_1 and p_2. There are two random variables, X_1 and X_2. The design parameters p_1 and p_2 are the means of the random variables X_1 and X_2, respectively. This problem involves a coupled system analysis and has two contributing analyses. The problem has two hard constraints:, g_1^R and g_2^R. The RBDO problem in standard form is as follows: Minimize

$$d_1^2 + 10d_2^2 + y_1$$

subject to

$$g_1^R = Y_1(\boldsymbol{X}, \eta)/8 - 1 \geq 0, \qquad g_2^R = 1 - Y_2(\boldsymbol{X}, \eta)/5 \geq 0$$

$$-10 \leq d_1 \leq 10, \qquad 0 \leq d_2 \leq 10$$

where

$$d_1 = \eta_1, \qquad d_2 = \eta_2, \qquad p_1 = \mu_{X_1} = 0, \qquad p_2 = \mu_{X_2} = 0$$

CA_1 is

$$Y_1(\boldsymbol{X}, \eta) = \eta_1^2 + \eta_2 - 0.2Y_2(\boldsymbol{X}, \eta) + X_1$$

$$y_1(\boldsymbol{d}, \boldsymbol{p}) = d_1^2 + d_2 - 0.2y_2(\boldsymbol{d}, \boldsymbol{p}) + p_1$$

and CA_2 is

$$Y_2(\boldsymbol{X}, \eta) = \eta_1 - \eta_2^2 + \sqrt{Y_1(\boldsymbol{X}, \eta)} + X_2$$

$$y_2(\boldsymbol{d}, \boldsymbol{p}) = d_1 - d_2^2 + \sqrt{y_1(\boldsymbol{d}, \boldsymbol{p})} + p_2$$

It is assumed that the random variables X_1 and X_2 have a uniform distribution over the intervals $[-1, 1]$ and $[-0.75, 0.75]$, respectively. The desired value of the reliability index β_{reqd_i} (for $i = 1, 2$) is chosen as 3 for both of the hard constraints.

Figure 4 shows the contours of the merit function and the constraints. The zero contours of the hard constraints are plotted at the design parameters p_1 and p_2 (mean of the random variables X_1 and

Table 2 Computational comparison of results (short rectangular column)

Formulation	f	$\partial f/\partial \boldsymbol{d}$	g^R	$\partial g^R/\partial \boldsymbol{d}$	$\partial g^R/\partial \boldsymbol{u}$
Nested (PMA)	8	8	77	8	77
Decoupled method	12	12	31	12	19

Table 1 Stochastic parameters in short column design problem

Variable	Symbol	Distribution	Mean/std. dev.	Unit	Correlations P	Correlations M	Correlations Y
Yield stress	P	Normal	500/100	MPa	1	0.5	0
Bending moments	M	Normal	2000/400	MNm	0.5	1	0
Axial force	Y	Lognormal	5/0.5	MPa	0	0	1

X_2). Note that in deterministic optimization, two local optima exist for this problem. At the global solution, only the first hard constraint is active, whereas at the local solution both of the hard constraints are active. They are shown by stars. Both of these solutions can be located easily by choosing different starting points in the design space.

Similarly, two local optimum designs exist for the RBDO problem as well. Both reliable designs get pushed into the feasible region, characterized with a higher merit function value and a lower probability of failure. They are shown by the shaded squares in Fig. 5.

To locate the two local optimal solutions of this problem, two different starting points, [−5, 3] and [5, 3], are chosen. The results corresponding to the starting point [−5, 3] are listed in Table 3.

Starting at the design $\boldsymbol{d} = [-5, 3]$ the proposed decoupled RBDO framework converges to the reliable optimum point without any difficulty. The proposed unilevel method requires 65 system analysis (SA) evaluations as compared to 225 when using the traditional double-loop PMA method. Analytical gradients were used in implementing this problem for all methods. Note that the double-loop method that uses the reliability index approach to prescribe the probabilistic constraints does not converge. For the designs that are visited by the upper-level optimizer, for example, $\boldsymbol{d}^k$ at the kth iteration, the FORM problem does not have a solution (because of zero failure probability at these designs). Starting from the design [−5, 3], the optimizer tries to find the local design [−3.006, 0.049]. However, it turns out that at this design, the second hard constraint g_2^R is never zero in the space of uniformly distributed random variables $\boldsymbol{X}$. Because in the RIA method the limit-state function is enforced as an equality constraint, the lower-level optimizer does not converge.

The results corresponding to the starting point [5, 3] are also listed in Table 3. Note that the double-loop method that uses the RIA for probabilistic constraint evaluation fails to converge for this starting point too. Again, the reason for this is that there is zero failure probability (infinite reliability index) at the designs visited by the upper-level optimizer and, therefore, the lower-level optimizer does not provide any true solution. All of the other methods converge to the same local optimum solution. The decoupled methodology developed in this investigation is found to be sufficiently efficient as compared to the nested formulation.

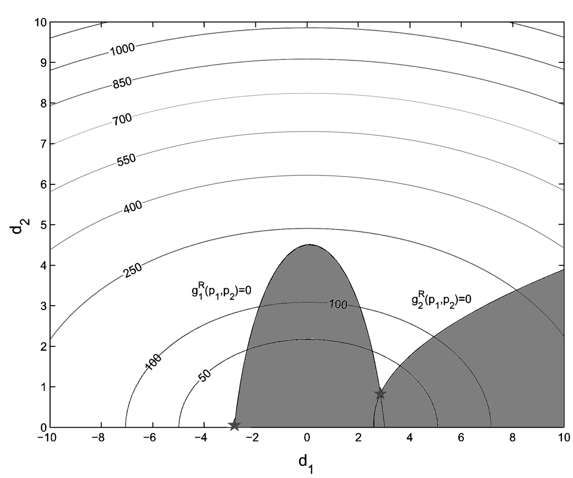

Fig. 4 Contours of objective and constraints.

Table 3 Starting points [−5, 3] and [5, 3]

Starting point	Cost measure	Double loop RIA	Double loop PMA	Decoupled RBDO
[−5, 3][a]	SA calls	Not converged	225	65
[5, 3][b]	SA calls	Not converged	184	65

[a]Solution [−3.006, 0.049]. [b]Solution [2.9277, 1.3426].

C. Cantilever Beam

This problem is taken from Ref. 4. A cantilever beam is subjected to an oscillatory fatigue loading Q_1 and random design load in service Q_2. The random variables in the problem are assumed to be independent with statistical parameters given in Table 4.

The design variables in the problem are width b and depth h of the beam. The objective is to minimize the weight of the beam bh (assuming unit weight per unit volume) subject to following hard constraints:

$$g_1^R = 0.3Eb^3d/900 - Q_2 \geq 0, \qquad g_2^R = A\left(6Q_1L\big/bd^2\right) - N_0 \geq 0$$

$$g_3^R = \Delta_0 - \left(4Q_2L^3\big/Ebd^3\right) \geq 0, \qquad g_4^R = R - \left(6Q_2L\big/bd^2\right) \geq 0$$

where $N_0 = 2 \times 10^6$, $\Delta_0 = 0.15$ in., and $L = 30$ in. A minimum reliability index of 3 is desired for each failure mode. It is clear that the beam design problem exhibits nonlinear limit-state functions (g_1^R–g_4^R), nonnormal random variables, and multicriteria constraints.

The optimization process was started from the point, $\boldsymbol{d}^0 = (1, 1)$. Both approaches result in an optimal solution $\boldsymbol{d}^* = (0.2941, 4.5559)$. The computational cost for the two methods is compared in terms of the total number of g-function evaluations taken by each method. The proposed decoupled RBDO method took 238 g-function evaluations as compared to 523 evaluations by the nested RBDO method. This does not include derivative calculations because analytical first-order derivatives were used. Therefore, note

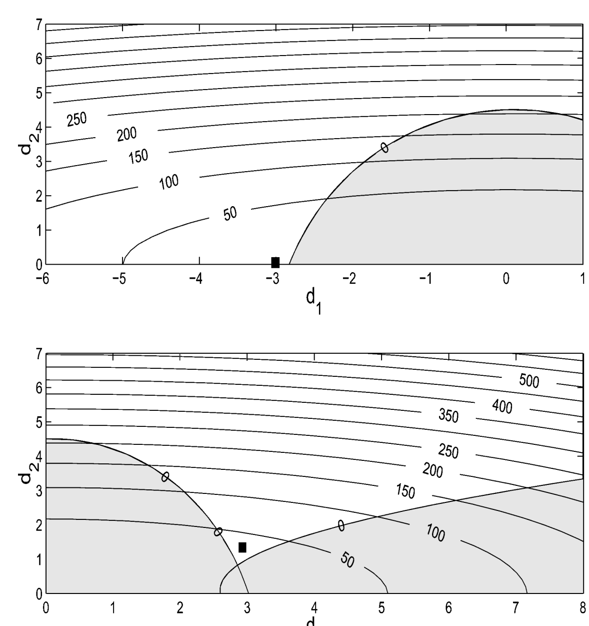

Fig. 5 Two reliable optima.

Table 4 Stochastic parameters in cantilever beam problem

Variable	Symbol	Distribution	Mean/std. dev.	Unit
Young's modulus	E	Normal	30,000/3,000 (206.85/20.685)	ksi (MPa)
Fatigue load	Q_1	Lognormal	0.5056/0.1492 (2.25/0.6637)	klb (N)
Random load	Q_2	Lognormal	0.4045/0.1492 (1.8/0.6637)	klb (N)
Unit yield strength	R	Weibull	50/6 (0.3448/0.0414)	ksi (MPa)
Fatigue strength coefficient	A	Lognormal	1.6323×10^{10}/0.4724 (1.10×10^8/0.0034)	ksi (MPa)

Table 5 Stochastic parameters in steel column problem

Variable	Symbol	Distribution	Mean/std. dev.	Unit
Yield stress	F_s	Lognormal	400/35	MPa
Dead weight load	P_1	Normal	500,000/50,000	N
Variable load	P_2	Gumbel	600,000/90,000	N
Variable load	P_3	Gumbel	600,000/90,000	N
Flange breadth	B	Lognormal	b/3	mm
Flange thickness	D	Lognormal	d/2	mm
Height of profile	H	Lognormal	h/5	mm
Initial deflection	F_0	Normal	30/10	mm
Young's modulus	E	Weibull	21,000/4,200	MPa

that the proposed methodology is significantly efficient compared to the traditional approach while providing the same solution.

D. Steel Column

This problem is taken from Kuschel and Rackwitz.[21] The problem is a steel column with design vector, $\boldsymbol{d} = (b, d, h)$, where b is the mean of flange breadth, d is the mean of flange thickness, and h is the mean of height of steel profile.

The length of the steel column s is 7500 mm. The objective is to minimize the cost function, $f = bd + 5h$. The independent random vector, $\boldsymbol{X} = (F_s, P_1, P_2, P_3, B, D, H, F_0, E)$, and its stochastic characteristics are given in Table 5.

The limit-state function in terms of the random vector $\boldsymbol{X}$, namely, the limit-state parameters $\boldsymbol{\eta} = \boldsymbol{d}$, is given as

$$G^R(\boldsymbol{X}, \boldsymbol{\eta}) = F_s - \mathcal{P}\{1/\mathcal{A}_s + (F_0/\mathcal{M}_s)[\varepsilon_b/(\varepsilon_b - \mathcal{P})]\}$$

where $\mathcal{A}_s$ is the $2BD$ area of section, $\mathcal{M}_s$ is the BDH modulus of section, $\mathcal{M}_i$ is the $\frac{1}{2}BDH^2$ moment of inertia, and ε_b is the $\pi^2 E\mathcal{M}_i/s^2$ (Euler buckling load). The means of the flange breadth b and flange thickness d must be within the intervals [200, 400] and [10, 30], respectively. The interval [100, 500] defines the admissible mean height h of the T-shaped steel profile. It is required that the optimal design satisfies a reliability level of three.

Again, both methods yield the same optimal solution, $\boldsymbol{d} = (200, 17.1831, 100)$. The computational cost of the two approaches is compared in terms of the number of g-function evaluations taken by each method. The proposed decoupled RBDO methodology took 236 evaluations of the limit-state function as compared to 457 evaluations taken by the nested RBDO approach. This does not include derivative calculations because analytical first-order derivatives were used. Again, note that the proposed methodology is significantly efficient compared to the traditional approach while providing the same solution.

VI. Conclusions

In this investigation, a new decoupled iterative RBDO methodology is developed. The deterministic optimization phase is separated from the reliability analysis phase. During the deterministic optimization phase the most probable point of failure corresponding to each failure mode is obtained by using a first-order Taylor series expansion about the design point from the preceding cycle. The MPP update during the deterministic optimization requires the sensitivities of the MPPs with respect to the design vector. These sensitivities are evaluated using postoptimality analysis techniques at the MPP optimal solution. Postoptimality analysis requires second-order derivatives of the failure mode. In this research, a damped BFGS update scheme is employed to compute the second-order derivatives at significantly reduced cost. Postoptimal sensitivities are used to account for the MPP coupling in the determinisitic optimization. It is observed that the estimated MPP converges to the exact values in a few cycles. The framework is tested using a series of structural and multidisciplinary design problems. It is found that the proposed methodology provides the same solution as the traditional nested optimization formulation and is significantly more efficient in computations required.

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